Fermion to Boson Mapping

Asis Basu and Jayprokas Chakrabarti

Theory Group

Indian Association for the Cultivation of Science

Calcutta-32 INDIA.

Abstract

The enveloping algebra,D_n,of fermions is extended on the lattice to include the discrete

space invariance.This extended algebra,denoted X, has the space symmetry as a factor :

$X/D_n =$ space group.

1

1 Introduction

In a recent couple of papers [1,2] we expressed the dynamics of fermions moving on the linear array of molecules in terms of bosonic bonds between the fermions. Boson realisations of Lie algebras have seen wide applications in many areas[3].Earlier it used to be in continuum theories, but now the goal is to go from continuum to lattice models[4] to solve them on the computer,for instance, in the quark-glue systems.The enveloping algebra for the case of spinless fermions on m-lattice sites is A_{2m-1} if it conserves the particle number.In the number non-conserving case it is D_{2m}.On the lattice both these have to extend to include the discrete space symmetry.

In this paper we extend our results [1,2] to two cases : i) when the simple translation by one lattice site is not the symmetry of the Hamiltonian and ii) in more than one dimensions as well as in arbitrary large dimensional systems.Our efforts, much like in [1,2] , is to transform the simple Hamiltonian of fermions to those of bond bosons.We indicate briefly how to write the fermionic interactions in terms of these bosons.

This mapping from the problem of fermions to the problem of bosons is motivated by the following reasons: a) In some situations, such as in the harmonic oscillator potential, the bosonic co-ordinates are physically more relevant[5].Some of the n-point fermion functions have poles that are bosonic.In these cases the theory below some threshold temperature is dominated by these bosons[3]. b) There are other cases where the fermion problem may be mapped to

some equivalent boson problem.This mapping could simplify the problem somewhat,for example,reducing the quartic fermion terms to quadratic in bosons etc.There have been wide uses of these types of boson-mapping in a variety of problems, both in condensed matter theory [6] as well as in other many-body systems[3].

There have been instances where the distinction between (a) and (b) disappears. The boson mapping of fermion theories generate approximate co-ordinates that are physically relevant. One such example is the Holstein-Primakoff mapping [7] of spin systems in terms of bosons.These bosons are the approximate physical degrees of freedom at least for high-spin Heisenberg interactions.In this paper we explore fermionic systems mapped to bosonic bonds. The mapping of fermions to bosons is a subject of interest on its own. As early as 1950 ,Tomonaga transformed the usual quartic term describing the interaction of electrons to terms in density fluctuations that were treated as bosons. The problem of interacting electron gas has been mapped to these bosons in one dimension [6] by Tomonaga , Luttinger , Mattis and others.In recent years the properties of fermions in low- dimensions have been a subject of great interest[8,9] .

The Fourier transform of the 2-point fermion correlation function gives the momentum distribution n(k).This quantity has a finite discontinuity at the fermi surface for fermi-liquids.This discontinuity is called the quasiparticle renormalisation factor z_k . In non-interacting theories

3

$z_k = 1$. As the interactions are turned on , $0 < z_k < 1$. The 2-point fermion function :

$$G(x - x', t - t') = < 0|T\psi(x,t)\psi^\dagger(x',t')|0 > \tag{1}$$

has the Fourier transform :

$$\widetilde{G}(k,\omega) = \frac{i}{\omega - \epsilon_k - \Sigma(k,\omega)} \tag{2}$$

where the chemical potential is absorbed in defining ϵ_k so that $\epsilon_k = 0$ for k on the fermi surface.The $\Sigma(k,\omega)$, called the self-energy, quantifies the effects of the interactions and also contains the prescription necessary to shift the pole off the real axis for ω . In case of fermi-liquids,

$$z_k = (1 - \frac{\partial \Sigma}{\partial \omega})^{-1} \tag{3}$$

gives the discontinuity of n(k) on the Fermi surface. It is known that fermion systems in low-dimensions, often behave otherwise.At the fermi-surface in these cases there is no discontinuity in the momentum distribution function.Instead there is a cusp whose form is determined from an exponent which depends on the interactions.This governs the power-law fall offs of the n-point functions as the space-time distances increase[10].

For the mapping of fermions to bosonic bonds we considered in [1,2] an one dimensional array of atoms/molecules on which the fermions move. This chain is assumed to have 2N sites.[It is not necessary to consider the site number to be even ; it could equally well be odd.] Consider

4

the bond variables :

$$e_{+l} = \sum c_n^\dagger c_{n+l}^\dagger \tag{4}$$

where $c_m^\dagger$ and c_m creates and destroys the fermion at site m. This m takes values from 1 to 2N. Clearly,

$$\{c_m, c_n^\dagger\} = \delta_{mn} \tag{5}$$

$$\{c_m, c_n\} = \{c_m^\dagger, c_n^\dagger\} = 0 \tag{6}$$

Note that in (4) we have paired fermions separated by l-sites and superposed. The reason for this superposition is to generate the object e_{+l} [1] which has translational invariance on the one-dimensional lattice. Our assumption here has been that the underlying dynamics i.e. the Hamiltonian ,is invariant under translation by one site. There are two points to keep in mind at this stage :

(1) that the object e_{+l} need not be invariant of the space group of the lattice.It is sufficient if it is an irreducible representation of that group.

(2) Even though the lattice has periodicities of translation by a site, the dynamics - the Hamiltonian - may not have that symmetry.In these cases it is convenient to define the bond variables with the underlying invariance.

Amongst the Vinyl Polymers Polyacetylene (PA) is a member made of $(CH)_X$. Each CH gives

one π- electron ; therefore the band is half-filled . The lattice-fermion interaction[11,12] makes this partially filled band unstable with respect to a lattice distortion at $2k_F$. ; k_F being the Fermi vector. Clearly, the Hamiltonian for this system is not invariant under translation by a single lattice site.Interestingly, the solutions of the Dirac Hamiltonian on the lattice are the long wavelength modes of the Hamiltonian for PA of Su-Schriffer-Heeger (SSH)[13,14]. We take up this case,define the boson-bond mapping and solve the SSH-Hamiltonian in terms of these bosons.

Coming back now to point (1) that the bosonic bonds need not be invariant with respect to the underlying space-group. To generate the ireducible representation we "distorted"[1,2] the superposed fermion pairs(l-sites apart) by defining the variables

$$e_{+lk} = \sum exp^{ikn}.c_n^\dagger c_{n+l}^\dagger \tag{7}$$

The quantities k are obtained using periodic boundary condition, $exp^{ik.2N} = 1$. Note we could have used the antiperiodic boundary condition as well. We showed in [1] and [2] that the quatities e_{+lk} along with their conjugates $e_{-lk} = (e_{+lk})^\dagger$ satisfy an approximate bosonic commutator relationship, when properly normalised :

$$[\frac{e_{+lk}}{\sqrt{2N}}, \frac{e_{-lk}}{\sqrt{2N}}] = \delta_{ll'}\delta_{kk'} \tag{8}$$

in the filling region where the fermion states are all nearly occupied . In this state e_{+lk} are the destruction operators of the bosonic bonds; e_{-lk} create them . The Hamiltonian of fermions

6

is transformed to the Hamiltonian of the bosonic bond co-ordinates $e_{\pm lk}$, much like what Kronig[15] did in terms of the bosonic density fluctuations.The transformed Hamiltonian of the bosonic-bond co-ordinates is then solved. We discussed these solutions in a few cases . The consistency of this approach is brought out by comparing the eigenstates of of bosonic bonds with respect to the solutions for the known fermionic problem.These comparisons convinced us of the reliability of the approximations.The procedure is extended here for the following cases:

a) The dynamics due to variety of reasons may not have the symmetry of the underlying lattice, i.e.periodicity w.r.t. translation by a single-site.We take up the well-understood case of PA in order to be able to compare our bond solutions to the known solutions in terms of the fermion.

b) to extend the considerations to higher dimensions. We define the bosonic bonds in higher dimensions and solve a case of fermionic Hamiltonian in terms of bosonic bonds in (2+1) D.

c) We discuss briefly the transformation of quartic fermionic interactions to quadratic terms in bosonic bonds.

2   Boson Mapping

Denote the algebra of the fermion bilinears by X.On the one-dimensional lattice of N-sites,these bilinears are of the type $c_i^\dagger c_j^\dagger$, $c_i^\dagger c_j$ and $c_i c_j$ (where i,j take values from 1 to N).These bilinears go into one another under the action of the discrete space invariance.For the N-point lattice,

for the spinless case,the algebra of the fermion bilinears is isomorphic to D_N. If we include spin it is D_{2N}. Clearly $D_{2N}(or D_N)$ is an invariant subalgebra of X because the action of the space group does not take the generators out of $D_{2N}(or D_N)$. Hence X/D_N= space group, for the spinless case;X/D_{2N}= space group, for the fermions with spin.The induced representations are the ones that become relevant as illustrated here.

Around 1958 Natta, Mazzanti and Corradini converted the monomer acetylene C_2H_2 into polymer PA. Because of the fermion-lattice interactions an energy gap develops at $\pm k_F$ and PA is a semiconductor.Sometimes the resulting distortion is called dimerisation that is the alternate short and long bonds.In our study we keep in mind these lattice distortions. The bond bosons we construct have to be irreducible representations of the space group of this distorted lattice.The monomers CH have six degrees of freedom.Only one of these , the translation along the chain direction is important in dimerisation.It is the one kept in the Hamiltonian.The fermion-lattice coupling in the tight-binding approximation,due to Ovchinnikov[11,12] , has a more general form than the Fröhlich Hamiltonian and reduces to it in the $k \to 0$ limit.This Hamiltonian was discussed by Su-Schrieffer-Heegger[13] to explore the soliton states in PA.
This Hamiltonian is a simple example of the case that it does not commute with translation by a single site :

$$H = -t_0 \sum (c_{n\sigma}^\dagger c_{(n+1)\sigma}) + 2\alpha.u \sum [(-1)^n . c_{n\sigma}^\dagger c_{(n+1)\sigma}] + h.c. \tag{9}$$

We construct the bond bosons to reflect the underlying lattice space group.Divide the lattice into odd(A) and the even(B) lattice and define the bond bosons as follows :

$$e_{+lk}^{A} = \sum_{n=odd} c_n^{\dagger} c_{n+l}^{\dagger} e^{ikn} \tag{10}$$

$$e_{+lk}^{A} = \sum_{n=even} c_n^{\dagger} c_{n+l}^{\dagger} e^{ik\frac{n}{2}} \tag{11}$$

The symbol $c_n^{\dagger}(c_n)$ denotes the creation(destruction) operator for the up-spin fermion at site n.The corresponding operators for spin down fermions at the same site are denoted by $d_n^{\dagger}(d_n)$ respectively. Similar bond- bosons with $c \rightarrow d$ (i.e. spin-up replaced by spin down fermions.) are:

$$\overline{e_{+lk}^{A}} = \sum_{n=odd} d_n^{\dagger} d_{n+l}^{\dagger} e^{ikn} \tag{12}$$

$$\overline{e_{+lk}^{B}} = \sum_{n=even} d_n^{\dagger} d_{n+l}^{\dagger} e^{ik\frac{n}{2}} \tag{13}$$

Note that in writing these operators we have used the idea that under translation by 2-lattice sites the $e_{lk}^{A(B)}$ goes over to $e_{lk}^{A(B)}$ upto an overall phase. This ensures that these are irreducible representations of the underlying space group.

The bond bosons listed above do not exhaust all the possibilities.The bonds between spin-up fermions and spin-down fermions are bosonic as well. These are listed hereunder:

$$e_{+lk}^{\prime A} = \sum_{n=odd} c_n^{\dagger} d_{n+l}^{\dagger} e^{ikn} \tag{14}$$

$$e_{+lk}^{\prime B} = \sum_{n=even} c_n^{\dagger} d_{n+l}^{\dagger} e^{ik\frac{n}{2}} \tag{15}$$

$$\overline{e}^{\prime A}_{+lk} = \sum_{n=odd} d^\dagger_n c^\dagger_{n+l} e^{ikn} \tag{16}$$

$$\overline{e}^{\prime B}_{+lk} = \sum_{n=even} d^\dagger_n c^\dagger_{n+l} e^{ik\frac{n}{2}} \tag{17}$$

We note that the SSH Hamiltonian is spin independent. That means that bosonic bonds could be between fermions of the same spin or between fermions of opposite spin. We shall discuss the physical relevance of these possibilities later in this section.

It is convenient to deal with the linear combinations defined as :

$$E^{A(B)}_{\pm lk}(\pm) = e^{A(B)}_{\pm lk} \pm \overline{e}^{A(B)}_{\pm lk} \tag{18}$$

$$D^{A(B)}_{\pm lk}(\pm) = e^{\prime A(B)}_{\pm lk} \pm \overline{e}^{\prime A(B)}_{\pm lk} \tag{19}$$

The subscript minus denotes the object that is hermitian conjugate of the one with subscript plus .

To calculate the dynamics of the bond bosons $E^{A(B)}_{\pm lk}(\pm)$ and $D^{A(B)}_{\pm lk}(\pm)$, we calculate the commutators with the Hamiltonian (9). We get:

$$[H, E^A_{+lk}] = E^A_{+(l+1)k}.t_{0,l+1} + E^A_{+(l-1)k}.t_{0,l} + E^B_{+(l+1)k}.t_{0,+}e^{ik} + E^B_{+(l-1)k}.t_{0,-} \tag{20}$$

$$[H, E^B_{+lk}] = E^B_{+(l+1)k}.t_{0,l} + E^B_{+(l-1)k}.t_{0,l+1} + E^A_{+(l+1)k}.t_{0,-} + E^A_{+(l-1)k}.t_{0,+}e^{-ik} \tag{21}$$

$$[H, D^A_{+lk}] = D^A_{+(l+1)k}.t_{0,l+1} + D^A_{+(l-1)k}.t_{0,l} + D^B_{+(l+1)k}.t_{0,+}e^{ik} + D^B_{+(l-1)k}.t_{0,-} \tag{22}$$

$$[H, D^B_{+lk}] = D^B_{+(l+1)k}.t_{0,l} + D^B_{+(l-1)k}.t_{0,l+1} + D^A_{+(l+1)k}.t_{0,-} + D^A_{+(l-1)k}.t_{0,+}e^{-ik} \tag{23}$$

10

where $t_{0,l} = t_0 + (-1)^l . 2\alpha u$ and $t_{0,\pm} = t_0 \pm 2\alpha u$ The commutators with the conjugates $E_{-lk}^{A(B)}(\pm)$ and $D_{-lk}^{A(B)}(\pm)$,may be obtained by taking the hermitian conjugates of the above four relations. We note that the bond objects $E_{\pm lk}^{A(B)}(\pm)$ and $D_{\pm lk}^{A(B)}(\pm)$ are bosonic.Using the bosonic commutator relations we can transform the Hamiltonian (9) written in terms of fermions into an equivalent Hamiltonian of bosonic bonds.

This equivalent bosonic bond Hamiltonian is :

$$H_B(E) = \sum_k [t_{0,l+1}(E_{+(l+1)k}^A E_{-lk}^A + E_{+lk}^A E_{-(l+1)k}^A) + t_{0,l}(E_{+(l+1)k}^B E_{-lk}^B + E_{+lk}^B E_{-(l+1)k}^B)]$$

$$+ [t_{0+}(e^{ik} E_{+(l+1)k}^B E_{-lk}^A + e^{-ik} E_{+lk}^A E_{-(l+1)k}^B) + t_{0-}(E_{+lk}^B E_{-(l+1)k}^A + E_{+(l+1)k}^A E_{-lk}^B)] \qquad (24)$$

We have in addition the terms of the D-sector . The Hamiltonian in the D-co-ordinates is denoted by $H_B(D)$.The subscript B is to emphasize that these are bosonic Hamiltonian. Now we want to diagonalize the complete bond bosonic Hamiltonian: $H_B = H_B(E) + H_B(D)$ To diagonalize $H_B(E)$ we go over to q-space by Fourier transformation.The procedure below for $H_B(E)$; for $H_B(D)$ it is similar. Define

$$E_{\pm lk}^{A(B)} = \frac{1}{\sqrt{L}} \sum E_{\pm qk}^{A(B)} . e^{\mp iql} \qquad (25)$$

Carrying out this Fourier transform over $H_B(E)$ we get

$$
\begin{aligned}
H_B(E) &= \sum_{q,k} [E^A_{+qk}.E^A_{-qk} + E^B_{+qk}.E^B_{-qk}]2t_0\cos q + \\
&\quad [E^A_{+qk}.E^A_{-(q-\pi)k} - E^B_{+qk}.E^B_{-(q-\pi)k}]4\alpha u i \sin q + \\
&\quad E^B_{+qk}.E^A_{-qk}[e^{\frac{ik}{2}}(2t_0\cos(k/2-q) + 4\alpha u i \sin(k/2-q))] \\
&\quad + E^A_{+qk}.E^B_{-qk}[e^{-\frac{ik}{2}}(2t_0\cos(k/2-q) - 4\alpha u i \sin(k/2-q))]
\end{aligned}
\tag{26}
$$

We now set $z = e^{\frac{ik}{2}}(2t_0\cos(k/2-q) + i.4\alpha u \sin(k/2-q))$. Therefore the Hamiltonian $H_B(E)$

to diagonalize becomes

$$
H_B(E) = \sum [E^A_{+qk}.E^A_{-qk} + E^B_{+qk}.E^B_{-qk}]y + [E^A_{+qk}.E^A_{-(q-\pi)k} - E^B_{+qk}.E^B_{-(q-\pi)k}]iX
$$

$$
+ E^B_{+qk}.E^A_{-qk}z + E^A_{+qk}.E^B_{-qk}\overline{z}
\tag{27}
$$

where $y = 2t_0\cos q$, $X = 4\alpha u \sin q$, $\overline{z}$ is complex conjugate of z. In matrix form :

$$
\begin{pmatrix}
y & -iX & \overline{z} & 0 \\
iX & -y & 0 & -\overline{z} \\
z & 0 & y & iX \\
0 & -z & -iX & -y
\end{pmatrix}
\tag{28}
$$

If we call the above matrix A , its eigenvalues λ are of interest. $A - \lambda.I = 0$ gives us the following

four values for λ:

$$
\lambda = \pm\sqrt{(2t_0\cos q)^2 + (4\alpha.u\sin q)^2} \mp \sqrt{(2t_0\cos(k/2-q))^2 + (4\alpha.u\sin(k/2-q))^2}
\tag{29}
$$

Let us now compare the above boson-bond spectrum with that of single fermions[13].The lattice-fermion interaction leads to a band gap , parametrized by $\triangle_K$, given by

$$\triangle_K = 4\alpha.u.\sin K \tag{30}$$

(Note that we have the lattice spacing a=1). The symbol K is to distinguish it from k that we have used for the boson case.The relation between k and K will be discussed shortly. The single fermion energy spectrum is given by

$$E_K = \pm\sqrt{\epsilon_K^2 + \triangle_K^2} \tag{31}$$

where

$$\epsilon_K = 2t_0 \cos K \tag{32}$$

with a,the lattice spacing is set to 1 and $\triangle_K$ given by (30).The upper spectra, the + sign in (31), is the conduction band ; the lower one the valence band.Before we compare this single-fermion spectrum with that of the bosons,eqn.(29), let us think in terms of a small PA chain $(CH)_6$ -i.e. the lattice of 6-sites.The values of k are obtained from $e^{3ik} = 1$. The values of q(eqn(25)) are obtained from the periodic boundary condition on the boson lattice[1].Since for the lattice of 6-points there are 3 independent values for l,the condition on q is:

$$e^{3iq} = 1 \tag{33}$$

13

For the 6-point $(CH)_6$ chain, the K values are $0, \frac{2\pi}{3}$ and $\frac{4\pi}{3}$.Thus K,k and q range over the same values. Looking now at the bosonic bond spectrum (29), we notice that depending on the choice of signs in (29),the bonds could be between two valence band fermions, or between two conduction band fermions or between a fermion in the valence band with another in the conduction band.The bond spectrum is in one to one correspondence with the single fermion spectrum. Hence the two descriptions are equivalent.This shows that the program that we have laid out for the construction of the bonds is consistent. The Hamiltonian $H_B(E) + H_B(D)$ is spin-independent.Therefore, the D-sector(i.e. bonds between the up and the down spins) is identical to the E-sector(i.e. bonds between spins of the same type)Diagonalisation of the D-sector gives the same result as (29).

3   In Higher Dimensions

We extend in this section the concept of bosonic bonds to higher dimensions. In the 1-dim. case we transformed the Hamiltonian of fermions to the Hamiltonian of bosonic bonds. In the previous section we discussed the invariant subalgebra D_n of X and the factor - the space invariance.The induced representations are important.We solved the Hamiltonian of bosonic bonds and compared with known solutions of the fermions to check for the consistency of our approach.The bosonic bonds satisfy the bosonic commutator rules; much like the density

fluctuations.We want to do the same for higher dimensions.The construction of the bonds involved pairing two fermions separated l-sites apart and superposing with similar other fermion pairs such that the resulting bond variable is an irreducible representation of the underlying space group.

To extend to two dimension consider the simple square lattice on which the fermions move. We define the bond boson operator as :

$$e_{+lm}(k_x, k_y) = \Sigma c^{\dagger}_{xy} c^{\dagger}_{x+l,y+m} exp^{i(xk_x+yk_y)} \tag{34}$$

along with its conjugate $e_{-lm}(k_x, k_y)$

The quantities k_x and k_y are determined by the p.b.c. The algebra of $e_{+lm}(k_x, k_y)$ closes [1] with another type of operators $h_{\pm ab}(k_x, k_y)$ defined as :

$$h_{+ab}(k_x, k_y) = \Sigma c^{\dagger}_{xy} c_{x+a,y+b} exp^{i(xk_x+yk_y)} \tag{35}$$

with $h_{-ab}(k_x, k_y) = (h_{ab}(k_x, k_y))^{\dagger}$ Using our arguments as earlier [1,2] it is then straightforward to establish that

$$[e_{+lm}(k_x, k_y), e_{-lm}(k_x, k_y)] = (2N)^2 \delta_{ll'} \delta_{mm'} \delta_{k_x k'_x} \delta_{k_y k'_y} \tag{36}$$

near the filling region where all the fermion states are occupied. Thus if we normalise as $e_{+lm}(k_x, k_y) \rightarrow \frac{1}{2N} e_{+lm}(k_x, k_y)$ these normalised bond bosons satisfy the standard bosonic commutators.

15

Setting up the Dirac Hamiltonian in two-dimension requires a two component wave function $\psi_1(\overrightarrow{x})$ at each site $\overrightarrow{x} = (x_1, x_2)$ defined as:

$$
\begin{pmatrix}
\psi_1(\overrightarrow{x}) \\
\psi_2(\overrightarrow{x})
\end{pmatrix}
\tag{37}
$$

The Hamiltonian with mass term

$$
H = \sum_{\overrightarrow{x}} i\psi^\dagger(\overrightarrow{x})(\alpha.\partial)\psi(\overrightarrow{x}) + \Delta\psi^\dagger(\overrightarrow{x})\beta\psi(\overrightarrow{x})
\tag{38}
$$

for the choice : $\alpha_1 = -\sigma_2$; $\alpha_2 = \sigma_1$; $\beta = \sigma_3$ can be written on the lattice as :

$$
H = \sum_{x,y} c_{xy}^\dagger([b_{x-1,y} - b_{x+1,y}] + i[b_{x,y+1} - b_{x,y-1}])
$$

$$
+ b_{xy}^\dagger([c_{x+1,y} - c_{x-1,y}] + i[c_{x,y+1} - c_{x,y-1}])
$$

$$
+ \Delta(c_{xy}^\dagger c_{xy} - b_{xy}^\dagger b_{xy})
\tag{39}
$$

where we have identified c_{xy} and b_{xy} with $\psi_1(\overrightarrow{x})$ and $\psi_2(\overrightarrow{x})$ respectively. Let us construct the operators:

$$
e^1_{+lm}(k) = \sum c_{xy}^\dagger c_{x+l,y+m}^\dagger exp^{i(k_x x + k_y y)}
\tag{40}
$$

$$
e^2_{+lm}(k) = \sum b_{xy}^\dagger b_{x+l,y+m}^\dagger exp^{i(k_x x + k_y y)}
\tag{41}
$$

$$
\overline{e}^1_{+lm}(k) = \sum c_{xy}^\dagger b_{x+l,y+m}^\dagger exp^{i(k_x x + k_y y)}
\tag{42}
$$

$$
\overline{e}^2_{+lm}(k) = \sum b_{xy}^\dagger c_{x+l,y+m}^\dagger exp^{i(k_x x + k_y y)}
\tag{43}
$$

16

Now , as earlier, build the linear combination

$$E_{\pm lm}^{\pm 1}(k) = \frac{1}{\sqrt{2}}[e_{\pm lm}^1(k) \pm e_{\pm lm}^2(k)] \tag{44}$$

$$E_{\pm lm}^{\pm 2}(k) = \frac{1}{\sqrt{2}}[\bar{e}_{\pm lm}^1(k) \pm \bar{e}_{\pm lm}^2(k)] \tag{45}$$

The $E_{\pm lm}$ operators satisfy bosonic commutator rules.

Let us now calculate the commutators of these operators with the the concerned Hamiltonian, so that we get the following successive results :

$$[H, E_{+lm}^{\pm 1}(k)] = E_{(l-1)m}^{\mp 2}(k)(1 \mp exp^{-ik_x}) \mp E_{(l+1)m}^{\mp 2}(k)(exp^{ik_x} \mp 1)$$

$$+iE_{l(m-1)}^{\pm 2}(k)(1 + exp^{-ik_y}) - iE_{l(m+1)}^{\pm 2}(k)(1 + exp^{ik_y}) - 2mE_{+lm}^{\mp 1}(k) \tag{46}$$

and

$$[H, E_{+lm}^{\pm 2}(k)] = E_{(l+1)m}^{\mp 1}(k)(1 \mp exp^{ik_x}) \mp E_{(l-1)m}^{\mp 1}(k)(1 \mp exp^{-ik_x})$$

$$+iE_{l(m-1)}^{\pm 1}(k)(1 + exp^{-ik_y}) - iE_{l(m+1)}^{\pm 1}(k)(1 - exp^{ik_y}) \tag{47}$$

From the above two commutator results the Hamiltonian for the bond bosons can be written as :

$$H_B = (1 - e^{ikx})[E_{+(l+1)m}^{-1}E_{-lm}^{+2} - E_{+(l+1)m}^{+2}E_{-lm}^{-1}]$$

$$+(1 + e^{ikx})[E_{+(l+1)m}^{+1}E_{-lm}^{-2} - E_{+(l+1)m}^{-2}E_{-lm}^{+1}]$$

17

$$+i(e^{iky} - 1)[E^{+1}_{+l(m+1)}E^{+2}_{-lm} - E^{+2}_{+l(m+1)}E^{+1}_{-lm}]$$

$$-i(e^{iky} + 1)[E^{-2}_{+l(m+1)}E^{-1}_{-lm} + E^{-1}_{+l(m+1)}E^{-2}_{-lm}]$$

$$-2mE^{-1}_{+lm}E^{+1}_{-lm} + h.c. \qquad (48)$$

We go over to Fourier transformed space(s-p) and define:

$$E^{\pm 1}_{\pm l,m} = \frac{1}{LM} \sum M^{\pm}_{\pm sp} e^{\mp isl} e^{\mp ipm} \qquad (49)$$

$$E^{\pm 2}_{\pm l,m} = \frac{1}{LM} \sum N^{\pm}_{\pm sp} e^{\mp isl} e^{\mp ipm} \qquad (50)$$

In terms of these new variables the equivalent Hamiltonian for this 2D case is:

$$H_B = \sum_{sp} N^{+}_{+sp} M^{-}_{-sp}[e^{is} - e^{-is} + e^{i(k_x - s)} - e^{-i(k_x - s)}]$$

$$+ M^{+}_{+sp} N^{-}_{-sp}[e^{-is} - e^{is} + e^{-i(k_x - s)} - e^{i(k_x - s)}]$$

$$+ iM^{+}_{+sp} N^{+}_{-sp}[e^{ip} - e^{-ip} + e^{i(k_y - p)} - e^{-i(k_y - p)}]$$

$$+ iN^{-}_{+sp} M^{-}_{-sp}[e^{ip} - e^{-ip} + e^{-i(k_y - p)} - e^{i(k_y - p)}]$$

$$-2m[M^{+}_{-sp} M^{-}_{+sp} + M^{-}_{-sp} M^{+}_{+sp}] \qquad (51)$$

If we set

$$S^{\pm} = \pm \sin s + \sin(k_x - s)$$

and

$$P^{\pm} = \mp \sin p + \sin(k_y - p)$$

18

then the above Hamiltonian can be written in the form

$$H_B = \sum_{sp} 2iS^+(N_+^+M_-^- - M_+^-N_-^+) + 2iS^-(M_+^+N_-^- - N_+^-M_-^+)$$

$$-2P^+(M_+^+N_-^+ - N_+^+M_-^+) + 2P^-(N_+^-M_-^- - M_+^-N_-^-)$$

$$-2m(M_-^+M_+^- + M_-^-M_+^+) \tag{52}$$

In matrix form we have:

$$\begin{pmatrix} 0 & -2m & -2P^+ & -2iS^- \\ -2m & 0 & 2iS^+ & 2P^- \\ -2P^+ & -2iS^+ & 0 & 0 \\ 2iS^- & 2P^- & 0 & 0 \end{pmatrix} \tag{53}$$

From the above matrix the the four eigenvalues λ can easily be obtained as:

$$\lambda = \mp\sqrt{m^2 + \sin^2 p + \sin^2 s} \pm \sqrt{m^2 + \sin^2(k_x - s) + \sin^2(k_y - p)} \tag{54}$$

Note that as in the previous case of PA , equation (54) is consistent with the well-known solutions of the Dirac Hamiltonian on the lattice for single fermions. The procedure above generalizes to arbitrary large dimensions.

4 Discussions

In the SSH case, eqn.(9), the lattice-fermion interactions along the chain led to bond-bosons that are irreducible of translations by two sites. The bond-boson eigenenergies, however ,are exact sums of energies of the component fermions.The mapping from fermions to bosons is energetically neutral.

So far we have discussed the mapping of fermions to bond-bosons for the hopping and for parts of fermion-lattice interactions.The fermion-fermion interactions,such as the coulomb ;

$$H_c = \frac{1}{2} \sum \alpha_{nm} (c_n^\dagger c_n)(c_m^\dagger c_m) \tag{55}$$

needs to be written in terms of the bond bosons.Rewrite it as:

$$H_c = -\frac{1}{2} \sum \alpha_{nm} (c_n^\dagger c_m^\dagger)(c_n c_m) \tag{56}$$

Now, since

$$e_{\pm lk} = \sum_p c_p^\dagger c_{p+l}^\dagger e^{\pm ipk} \tag{57}$$

Therefore ,

$$c_p^\dagger c_{p+l}^\dagger = \sum_k e_{+lk} e^{-ipk} \tag{58}$$

.Hence all bilinears of the type $c_m^\dagger c_n^\dagger$ or $c_m c_n$ may be expressed in terms of the bosonic translational eigenstates $e_{\pm lk}$. Hence the interaction H_c is expressible in terms of the bond-bosons.Aside from the coulomb ,many other interactions may be mapped to bond-bosons by

the above method. The mapping yields a quadratic in boson-bonds in place of the quartic fermions. Similarly using the previous relations we can convert an interaction term of the type $\sum c_i^\dagger c_i(b_i + b_i^\dagger)$, where b_i are quantized lattice vibrations, into a form that has interactions between b_i with bond-bosons. The fermion gauge-boson interactions $\sum c_n^\dagger c_m e^{i. \int_n^m A.dl}$ can be reduced in the same manner to interactions between the gauge-bosons and the bond-bosons.

It is known from previous work[3] that the enveloping algebra for fermions for the number-conserving case is A_n. It is also known that the fermions belong to the fundamental (totally antisymmetric) representations of A_n For the lattice the number of fermion states are determined by the number of lattice points. The filling-factor determines, for this case , the specific fundamental representation to which the fermions belong. The lattice determines, to a large extent, the ways to combine the generators of the enveloping algebra for it to be useful. For instance the Cartan generators have to be suitably combined as follows: the "distorted" generators h_{ik} [1] do not commute with each other. These have to be combined to construct hermitian operators that constitute the Cartan subalgebra with the right space-group properties to be useful on the lattice. Denote the enveloping algebra on the lattice for the number conserving case by Y. It has A_n extended to include the discrete space invariance. The induced representations are relevant. The physically interesting boson mapping is constrained by the structure:

Y/A_N=Discrete space symmetry

21

In case the underlying dynamics does not conserve the number,it is known that the envelope algebra is D_n ; the fermions belong to the two irreducible spinor representations.The fundamental representations of A_n embed in these two spinor representations of D_n. Once again the lattice extends the algebra beyond D_n to X in order to include the discrete space invariance, with the constraint that D_n is an invariant subalgebra of X and that X/D_n=the discrete space symmetry

The generators have to be suitably combined to identify physically interesting boson degrees.These depend on the underlying space group.The physically interesting boson-bond configurations for the SSH-case was to illustrate some of these ideas.

As we go beyond 1+1 dimension into higher dimensions these ideas are generalized. Once again, the underlying lattice space invariance dictates how to combine the elements of the algebra.The illustration in section 3 is but one such.In general the useful combinations of generators that are bosonic depend on the space symmetry. The induced representations are the ones that are important.

5   Acknowledgements

References

[1] J.Chakrabarti,A.Basu and B.Bagchi, Phys.Lett.B500(2001)105.

[2] J.Chakrabarti,A.Basu and B.Bagchi, Europhys.Lett.58(2002)489.

[3] K.Sawada, Phys. Rev. 106(1957)372; S.T.Belyaev and V.G.Zelevinsky, Nucl. Phys. 39(1962)582; T.Marumori, M.Yamamura and A.Tokunaga, Prog. Theor. Phys. 31(1964)1009; H.J.Lipkin, N.Meshkov and A.Glick, Nucl.Phys.62(1965)188; Y.Gambhir, P.Ring and P.Schuck, Nucl. Phys A423(1984)35; W.M.Zhang, D.H.Feng and R.Gilmore, Rev. Mod. Phys.62(1990)867; A.Klein and E.R.Marshalek, Rev. Mod. Phys 63(1991)375.

[4] J.Smit, Introduction to Quantum Fields on a Lattice; Cambridge University Press, Cambridge, 2002; M.Creutz, Quantum Fields on the Computers, World Scientific, Singapore 1992.

[5] S.Okubo,Phys.Rev.C10(1974)2048; E.R.Marshalek,Phys.Rev. C11(1975)1426.

[6] S.T.Tomonaga, Prog. Theor. Phys 5(1950)544; J.M.Luttinger, J.Math.Phys. 4(1963)1154; D.C.Mattis and E.H.Lieb, J. Math. Phys. 6(1965)304; A.O.Gogolin, A.A.Nersesyan and A.M.Tsvelik, Bosonisation and Strongly Correlated Systems(Cambridge Univ.Press,1998).

[7] T.Holstein and H.Primakoff, Phys.Rev.58(1940)1098.

[8] F.D.M.Haldane,Phys.Rev.Lett.45(1980)1358,ibid.47(1981)1840

[9] H.J.Schultz in Proceedings of Les Houches Summer School LXI, edited by E.Akkermans, G.Montambaux, J.Pichard, and J.Zinn-Justin (Elsevier, Amsterdam, 1995)

[10] A.V.Zabrodin and A.A.Ovchinnikov, Sov.Phys.JETP 55(1982)162; V.Ya Krivnov and A.A.Ovchinnikov, Sov.Phys.JETP 63(1986)1326.

[11] A.A.Ovchinnikov , Sov.Phys.Solid State 7(1965)664.

[12] S.Barisic, Phys.Rev. B5(1972)932.

[13] S.A.Brazovskii, Sov.Phys. JETP. Lett. 28(1978)606; W.P.Su,J.R.Schrieffer and A.J.Heeger, Phys.Rev.Lett. 42(1979)1698; M.J.Rice, Phys.Lett. 71A(1979)152.

[14] Y.Onodera and S.Okuno, J. Phys.Soc.Jpn. 52(1983)2478; H.Takayama, Y.R.Lin-Liu. and K.Maki, Phys.Rev. B21(1980)2388; D.K.Campbell. and A.R.Bishop,Phys.Rev.B24(1981)4859.

[15] R.De.L.Kronig, Physica II 491(1935)854;968.